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Short Communication

On the construction of the solution of an equation describing an axially moving string

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Abstract

In this paper an initial boundary value problem for a linear, nonhomogeneous axially moving string equation will be considered. The velocity of the string is assumed to be constant, and the nonhomogeneous terms in the string equation are due to external forces acting on the string. The Laplace transform method will be used to construct the solution of the problem. It will turn out that the method has considerable, computational advantages compared to the usually applied method of modal analysis based on eigenfunction expansions.

1. Introduction

The dynamics of axially moving materials have been studied by many researchers due to their technological importance. Transversal vibrations of belt systems have been investigated for many years. A lot of literature is devoted to this problem (see the reference lists in Refs. [1–5]). In this paper a linear, nonhomogeneous equation for a moving string will be studied. The main goal of this paper is to study the effectiveness of the Laplace transform method with respect to the classical modal approach for these type of equations. The displacement of the moving string in

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vertical direction will be determined by using the Laplace transform method. The solution for this problem can be also constructed by using the method of eigenfunction expansions. This method was developed for these types of problems by Meirovitch [6,7] and by Wickert and Mote [1]. Both methods will be compared. The conditions under which bounded solutions exist will be derived, and it will be shown for what kind of external forces internal resonances in the system will occur.

The following linear equation of motion for the string (moving in one direction with a constant velocity V_0) will be considered in this paper:

$$u_{tt} + 2V_0 u_{xt} + (V_0^2 - c^2)u_{xx} = g(x,t), \quad 0<x<l, \ \ t>0, \tag{1}$$

where $u(x,t)$ is the displacement of the string in the vertical direction, V_0 the string speed, c the wave speed, x the coordinate in horizontal direction, $g(x,t)$ the external force, t the time, and l the distance between the pulleys.

In this paper the case $V_0<c$ is considered. At the pulleys it is assumed that there is no displacement of the string in vertical direction. Eq. (1) can also be found in Ref. [2], but now it is assumed that V_0 is not necessarily small. The boundary and initial conditions for $u(x,t)$ are given by

$$\begin{aligned} &u(0,t) = u(l,t) = 0, \quad t \geqslant 0, \\ &u(x,0) = f(x), \quad \text{and} \quad u_t(x,0) = r(x), \quad 0<x<l, \end{aligned} \tag{2}$$

where $f(x)$ and $r(x)$ represent the initial displacement and the initial velocity of the string, respectively. It is assumed that the functions $f(x)$ and $r(x)$ are sufficiently smooth such that a two times continuously differentiable solution for the initial boundary value problem (1)–(2) exists.

2. Application of the Laplace transform method

The initial boundary value problem (1)–(2) for $u(x,t)$ can readily be solved by applying the Laplace transform method (with respect to time t) to Eqs. (1)–(2), yielding:

$$\begin{aligned} &s^2U(x,s) - su(x,0) - u_t(x,0) + 2V_0(sU_x(x,s) - u_x(x,0)) \\ &\quad + U_{xx}(x,s)(V_0^2 - c^2) = G_1(x,s), \end{aligned} \tag{3}$$

$$U(0,s) = U(l,s) = 0, \tag{4}$$

where $U(x,s)$ and $G_1(x,s)$ are the Laplace transforms of $u(x,t)$ and $g(x,t)$, respectively.

By dividing Eq. (3) by $(V_0^2 - c^2)$ and by rearranging terms in Eq. (3) it follows that:

$$U_{xx} + \frac{2V_0 s}{V_0^2 - c^2}\,U_x + \frac{s^2}{V_0^2 - c^2}\,U = G(x,s), \tag{5}$$

where

$$G(x,s) = \frac{G_1(x,s) + sf(x) + r(x) + 2V_0 f_x(x)}{V_0^2 - c^2}. \tag{6}$$

The general solution of the homogeneous equation (that is, Eq. (5) with $G \equiv 0$) is given by

$$U(x,s) = C_1(s)\exp\left(\frac{-sx}{V_0+c}\right) + C_2(s)\exp\left(\frac{-sx}{V_0-c}\right),$$

where $C_1(s)$ and $C_2(s)$ are still arbitrary functions. The method of variation of parameters can be used to find the particular solution of the nonhomogeneous equation (5). By using this method the solution is found in the form:

$$U(x,s) = C_1(x,s)\exp\left(\frac{-sx}{V_0+c}\right) + C_2(x,s)\exp\left(\frac{-sx}{V_0-c}\right), \tag{7}$$

where $C_1(x,s)$ and $C_2(x,s)$ are given by

$$C_1(x,s) = \frac{(V_0^2-c^2)}{2sc}\int_0^x G(x^*,s)\exp\left(\frac{x^*s}{V_0+c}\right)\mathrm{d}x^* + K_1(s),$$

$$C_2(x,s) = -\frac{(V_0^2-c^2)}{2sc}\int_0^x G(x^*,s)\exp\left(\frac{x^*s}{V_0-c}\right)\mathrm{d}x^* + K_2(s),$$

where $K_1(s)$ and $K_2(s)$ are still arbitrary functions. The solution of the nonhomogeneous equation (3) or (5) is given by

$$\begin{aligned} U(x,s) = {} & K_1(s)\exp\left(\frac{-sx}{V_0+c}\right) + K_2(s)\exp\left(\frac{-sx}{V_0-c}\right) \\ & + \frac{(V_0^2-c^2)}{2sc}\int_0^x G(x^*,s)\left(\exp\left(\frac{-s(x-x^*)}{V_0+c}\right) - \exp\left(\frac{-s(x-x^*)}{V_0-c}\right)\right)\mathrm{d}x^*, \end{aligned} \tag{8}$$

where $K_1(s)$ and $K_2(s)$ can be determined from the boundary conditions (4). So, finally the following expression for $U(x,s)$ is found:

$$\begin{aligned} U(x,s) = {} & \frac{(V_0^2-c^2)}{2sc}\,\frac{\int_0^l G(x^*,s)\left(\exp\left(\frac{-s(l-x^*)}{V_0+c}\right) - \exp\left(\frac{-s(l-x^*)}{V_0-c}\right)\right)\mathrm{d}x^*}{\exp\left(\frac{-sl}{V_0+c}\right) - \exp\left(\frac{-sl}{V_0-c}\right)} \\ & \times\left(-\exp\left(\frac{-sx}{V_0+c}\right) + \exp\left(\frac{-sx}{V_0-c}\right)\right) \\ & + \frac{(V_0^2-c^2)}{2sc}\int_0^x G(x^*,s)\left(\exp\left(\frac{-s(x-x^*)}{V_0+c}\right) - \exp\left(\frac{-s(x-x^*)}{V_0-c}\right)\right)\mathrm{d}x^*. \end{aligned} \tag{9}$$

The inverse Laplace transform of $U(x,s)$ is given by

$$u(x,t) = \frac{1}{2\pi\mathrm{i}}\int_{v-\mathrm{i}\infty}^{v+\mathrm{i}\infty} U(x,s)\mathrm{e}^{st}\,\mathrm{d}s = \sum_n \mathrm{Res}(s_n,x,t) \quad \text{for some } v>0 \tag{10}$$

and where Res stands for the residue at $s = s_n$. To evaluate the inverse Laplace transform (10) the poles of $U(x,s)$ and the order of these poles have to be determined in the complex s-plane. As long as $g(x,t)$ is not specified the poles due to $G_1(x,s)$ will be unknown. For that reason three cases will be considered: (i) $g(x,t) = 0$, (ii) $g(x,t) = \varphi(x)\sin(\omega t)$ with $\omega = (\pi n^*/lc)(V_0^2 - c^2)$ for some fixed

$n^* \in \mathbb{Z}$, and (iii) $g(x,t) = \varphi(x)\sin(\omega t)$ with ω is not in the neighborhood of $(\pi n/lc)(V_0^2 - c^2)$ for all $n \in \mathbb{Z}$.

2.1. Case (i): $g(x,t) = 0$

In this case the poles of $U(x,s)$ follow from (see Eq. (9))

$$s\left(\exp\left(\frac{-sl}{V_0+c}\right) - \exp\left(\frac{-sl}{V_0-c}\right)\right) = 0. \tag{11}$$

Now it should be observed that $s = 0$ is not a pole of $U(x,s)$ since $\lim_{s\to 0} U(x,s)$ exists. All other poles of $U(x,s)$ now follow from

$$\exp\left(\frac{-sl}{V_0+c}\right) - \exp\left(\frac{-sl}{V_0-c}\right) = 0$$

and are given by

$$s_n = \frac{\pi n}{lc}(V_0^2 - c^2)\mathrm{i}, \tag{12}$$

with $n \in \mathbb{Z}\backslash\{0\}$. It should be observed that these poles are all simple. The solution of the initial boundary value problem (1)–(2) with $g(x,t) = 0$ now easily follows from Eq. (10), yielding

$$\begin{aligned} u(x,t) = \sum_{n=1}^{\infty}\Bigg\{ & a_n\left(\cos\left(\frac{\pi n(V_0^2-c^2)t}{lc}\right)\left(\cos\left(\frac{\pi n(V_0+c)x}{lc}\right) - \cos\left(\frac{\pi n(V_0-c)x}{lc}\right)\right)\right. \\ & \left. + \sin\left(\frac{\pi n(V_0^2-c^2)t}{lc}\right)\left(\sin\left(\frac{\pi n(V_0+c)x}{lc}\right) - \sin\left(\frac{\pi n(V_0-c)x}{lc}\right)\right)\right) \\ & + b_n\left(\cos\left(\frac{\pi n(V_0^2-c^2)t}{lc}\right)\left(\sin\left(\frac{\pi n(V_0+c)x}{lc}\right) - \sin\left(\frac{\pi n(V_0-c)x}{lc}\right)\right)\right. \\ & \left. - \sin\left(\frac{\pi n(V_0^2-c^2)t}{lc}\right)\left(\cos\left(\frac{\pi n(V_0+c)x}{lc}\right) - \cos\left(\frac{\pi n(V_0-c)x}{lc}\right)\right)\right)\Bigg\}, \end{aligned}$$

where

$$\begin{aligned} a_n = \frac{(V_0^2-c^2)}{c}\Bigg(& \frac{1}{2lc}\int_0^l f(x^*)\left(\cos\left(\frac{(V_0-c)\pi n x^*}{lc}\right) - \cos\left(\frac{(V_0+c)\pi n x^*}{lc}\right)\right)\mathrm{d}x^* \\ & + \frac{1}{2}\int_0^l \frac{r(x^*) + 2V_0 f_x(x^*)}{\pi n(V_0^2-c^2)}\left(\sin\left(\frac{(V_0-c)\pi n x^*}{lc}\right) - \sin\left(\frac{(V_0+c)\pi n x^*}{lc}\right)\right)\mathrm{d}x^*\Bigg), \end{aligned}$$

$$\begin{aligned} b_n = \frac{(V_0^2-c^2)}{c}\Bigg(& \frac{1}{2lc}\int_0^l f(x^*)\left(\sin\left(\frac{(V_0-c)\pi n x^*}{lc}\right) - \sin\left(\frac{(V_0+c)\pi n x^*}{lc}\right)\right)\mathrm{d}x^*. \\ & - \frac{1}{2}\int_0^l \frac{r(x^*) + 2V_0 f_x(x^*)}{\pi n(V_0^2-c^2)}\left(\cos\left(\frac{(V_0-c)\pi n x^*}{lc}\right) - \cos\left(\frac{(V_0+c)\pi n x^*}{lc}\right)\right)\mathrm{d}x^*\Bigg). \end{aligned}$$

2.2. Case (ii): $g(x,t) = \varphi(x)\sin(\omega t)$, the resonant case

In this case it is assumed that ω is equal to an eigenfrequency of the axially moving string (Fig. 1), that is, $\omega = (\pi n^*/lc)(V_0^2 - c^2)$ for some fixed $n^* \in \mathbb{Z}$. The Laplace transform of $g(x,t)$ in this case is

$$G_1(x,s) = \frac{\omega}{s^2+\omega^2}\,\varphi(x).$$

In Eq. (9) $G(x,s)$ is now given by $G(x^*,s) = \omega\varphi(x^*)/((s^2+\omega^2)(V_0^2-c^2)) + h(x^*,s)$, where $h(x^*,s) = (sf(x^*) + r(x^*) + 2V_0 f_{x^*}(x^*))/(V_0^2-c^2)$. And so, $U(x,s)$ can be written as $U(x,s) = A(x,s) + B(x,s) + D(x,s) + E(x,s)$, where

$$A(x,s) = \frac{(V_0^2-c^2)}{2sc}\,\frac{\omega\int_0^l \varphi(x^*)\left(\exp\left(\frac{-s(l-x^*)}{V_0+c}\right) - \exp\left(\frac{-s(l-x^*)}{V_0-c}\right)\right)\mathrm{d}x^*}{(s^2+\omega^2)(V_0^2-c^2)\left(\exp\left(\frac{-sl}{V_0+c}\right) - \exp\left(\frac{-sl}{V_0-c}\right)\right)} \times\left(-\exp\left(\frac{-sx}{V_0+c}\right) + \exp\left(\frac{-sx}{V_0-c}\right)\right), \tag{13}$$

$$B(x,s) = \frac{(V_0^2-c^2)}{2sc}\,\frac{\int_0^l h(x^*,s)\left(\exp\left(\frac{-s(l-x^*)}{V_0+c}\right) - \exp\left(\frac{-s(l-x^*)}{V_0-c}\right)\right)\mathrm{d}x^*}{\left(\exp\left(\frac{-sl}{V_0+c}\right) - \exp\left(\frac{-sl}{V_0-c}\right)\right)} \times\left(-\exp\left(\frac{-sx}{V_0+c}\right) + \exp\left(\frac{-sx}{V_0-c}\right)\right), \tag{14}$$

$$D(x,s) = \frac{\omega}{2sc(s^2+\omega^2)}\int_0^x \varphi(x^*)\left(\exp\left(\frac{-s(x-x^*)}{V_0+c}\right) - \exp\left(\frac{-s(x-x^*)}{V_0-c}\right)\right)\mathrm{d}x^*, \tag{15}$$

$$E(x,s) = \frac{(V_0^2-c^2)}{2sc}\int_0^x h(x^*,s)\left(\exp\left(\frac{-s(x-x^*)}{V_0+c}\right) - \exp\left(\frac{-s(x-x^*)}{V_0-c}\right)\right)\mathrm{d}x^*. \tag{16}$$

The inverse Laplace transform of $U(x,s)$ is given by

$$L^{\mathrm{inv}}(U(x,s)) = L^{\mathrm{inv}}(A(x,s)) + L^{\mathrm{inv}}(B(x,s)) + L^{\mathrm{inv}}(D(x,s)) + L^{\mathrm{inv}}(E(x,s)). \tag{17}$$

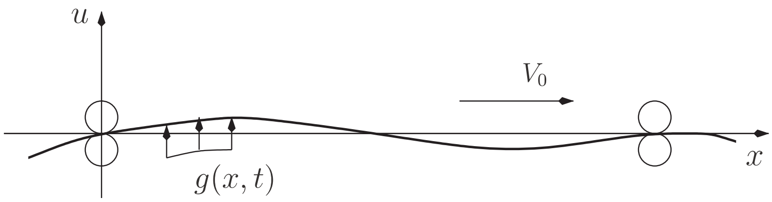

Fig. 1. An axially moving string.

In Eq. (17), $L^{\text{inv}}(B(x,s))$ and $L^{\text{inv}}(E(x,s))$ only depend on the initial values $f(x)$ and $r(x)$ and already have been determined in case (i) with $g(x,t)=0$. So, only $L^{\text{inv}}(A(x,s))$ and $L^{\text{inv}}(D(x,s))$ have to be calculated. It should be observed that these inverse Laplace transforms only depend on $g(x,t)$. Furthermore, it should be observed that $A(x,s)$ is the product of $\omega/(s^2+\omega^2)$ and another term (following from Eq. (13)), and so the inverse Laplace transform of $A(x,s)$ can be determined by using the convolution integral, that is,

$$L^{\text{inv}}(A(x,s)) = \int_0^t \sin(\omega(t-\tau)) \sum_n \operatorname{Res}(s_n, x, \tau)\, \mathrm{d}\tau, \tag{18}$$

where s_n is given by Eq. (12). Finally, if $\omega = (\pi n^*/lc)(V_0^2 - c^2)$ for a fixed $n^* \in \mathbb{Z}$ it follows that $L^{\text{inv}}(A(x,s))$ is

$$\begin{aligned} L^{\text{inv}}(A(x,s)) = {} & \frac{F_{1n^*}(x)}{2}\, t\, \sin(\omega_{n^*} t) + \frac{F_{2n^*}(x)}{2}\left(\frac{\sin(\omega_{n^*} t)}{\omega_{n^*}} - t\cos(\omega_{n^*} t)\right) \\ & + \sum_{n=1, n\neq |n^*|}^{\infty} \left(\frac{F_{1n}(x)}{2}\left(\frac{1}{\omega_{n^*} - \omega_n}(\cos(\omega_n t) - \cos(\omega_{n^*} t))\right.\right. \\ & \left. + \frac{1}{\omega_{n^*} + \omega_n}(\cos(\omega_n t) - \cos(\omega_{n^*} t))\right) \\ & + \frac{F_{2n}(x)}{2}\left(\frac{1}{\omega_{n^*} + \omega_n}(\sin(\omega_n t) + \sin(\omega_{n^*} t))\right. \\ & \left.\left. + \frac{1}{\omega_{n^*} - \omega_n}(\sin(\omega_n t) - \sin(\omega_{n^*} t))\right)\right), \end{aligned} \tag{19}$$

where

$$\begin{aligned} F_{1n}(x) = {} & w_n\left(\cos\left(\frac{\pi n(V_0+c)x}{lc}\right) - \cos\left(\frac{\pi n(V_0-c)x}{lc}\right)\right) \\ & + p_n\left(\sin\left(\frac{\pi n(V_0+c)x}{lc}\right) - \sin\left(\frac{\pi n(V_0-c)x}{lc}\right)\right), \end{aligned} \tag{20}$$

$$\begin{aligned} F_{2n}(x) = {} & w_n\left(\sin\left(\frac{\pi n(V_0+c)x}{lc}\right) - \sin\left(\frac{\pi n(V_0-c)x}{lc}\right)\right) \\ & - p_n\left(\cos\left(\frac{\pi n(V_0+c)x}{lc}\right) - \cos\left(\frac{\pi n(V_0-c)x}{lc}\right)\right), \end{aligned} \tag{21}$$

$$w_n = \frac{1}{2c}\int_0^l \frac{\varphi(x^*)}{\pi n}\left(\sin\left(\frac{(V_0-c)\pi n x^*}{lc}\right) - \sin\left(\frac{(V_0+c)\pi n x^*}{lc}\right)\right)\mathrm{d}x^*, \tag{22}$$

$$p_n = -\frac{1}{2c}\int_0^l \frac{\varphi(x^*)}{\pi n}\left(\cos\left(\frac{(V_0-c)\pi n x^*}{lc}\right) - \cos\left(\frac{(V_0+c)\pi n x^*}{lc}\right)\right)\mathrm{d}x^*. \tag{23}$$

It can be seen from Eq. (13) that if $n = n^*$ then there are two poles of order two (one in $s = \mathrm{i}\omega_{n^*}$, and one in $s = -\mathrm{i}\omega_{n^*}$). To calculate $L^{\mathrm{inv}}(D(x,s))$ it should be observed that $s = 0$ is not a pole as $\lim_{s\to 0} D(x,s)$ exists. So the inverse Laplace transform for $D(x,s)$ is

$$L^{\mathrm{inv}}(D(x,s)) = \frac{(V_0^2 - c^2)}{c\omega}\left(\frac{1}{2}\cos(\omega t)\int_0^x \varphi(x^*)\left(\cos\left(\frac{\omega(x-x^*)}{V_0+c}\right) - \cos\left(\frac{\omega(x-x^*)}{V_0-c}\right)\right)\mathrm{d}x^* \right.$$
$$\left. + \frac{1}{2}\sin(\omega t)\int_0^x \varphi(x^*)\left(\sin\left(\frac{\omega(x-x^*)}{V_0+c}\right) - \sin\left(\frac{\omega(x-x^*)}{V_0-c}\right)\right)\mathrm{d}x^*\right). \tag{24}$$

It can be seen that only $L^{\mathrm{inv}}(A(x,s))$ contributes to unbounded terms in the solution.

2.3. Case (iii): $g(x,t) = \varphi(x)\sin(\omega t)$, *the nonresonant case*

Let $\omega_n = (\pi n/lc)(V_0^2 - c^2)$, $n \in \mathbb{Z}$ be the natural frequencies of an axially moving string and let ω be not in a neighborhood of any of these frequencies ω_n.

For $L^{\mathrm{inv}}(A(x,s))$ it then follows that

$$L^{\mathrm{inv}}(A(x,s)) = \sum_{n=1}^{\infty}\left(\frac{F_{1n}(x)}{2}\left(\frac{1}{\omega-\omega_n}(\cos(\omega_n t) - \cos(\omega t)) + \frac{1}{\omega+\omega_n}(\cos(\omega_n t) - \cos(\omega t))\right)\right.$$
$$\left. + \frac{F_{2n}(x)}{2}\left(\frac{1}{\omega+\omega_n}(\sin(\omega_n t) + \sin(\omega t)) + \frac{1}{\omega-\omega_n}(\sin(\omega_n t) - \sin(\omega t))\right)\right), \tag{25}$$

where $F_{1n}(x)$ and $F_{2n}(x)$ are given by Eqs. (20) and (21), respectively. $L^{\mathrm{inv}}(D(x,s))$ is again given by Eq. (24). In this case there are no unbounded terms in the solution $u(x,t)$. Obviously, unbounded solutions will occur when $g(x,t)$ contains terms $\varphi_1(x)\sin(\omega t)$ and/or terms $\varphi_2(x)\cos(\omega t)$ for which ω is equal to an eigenfrequency ω_n.

3. Conclusions and remarks

In this paper an initial boundary value problem for a linear, nonhomogeneous axially moving string equation has been studied. The velocity of the string is assumed to be constant, and the nonhomogeneous terms in the string equation are due to external forces acting on the string. To solve the initial boundary value problem the Laplace transform method had been used in this paper. There is also another, well-known method to solve this problem which is the modal analysis based on eigenfunction expansions. This method has been introduced in Refs. [1,6,7], and is used nowadays frequently for these types of problems (see for instance Refs. [3,4]). To apply this method an operator notation has to be introduced, an inner product has to be defined, an eigenvalue problem has to be solved, and orthonormality relations have to be determined. Altogether this method is rather complicated to apply to these types of problems. For that reason in this paper it is proposed to apply the Laplace transform method to these types of problems. When this method is applied poles (and the order of the poles) have to be determined, residues have to be calculated, and Cauchy's theorem has to be used (that is, integrals have to be evaluated by using the theory of complex variables). To construct the solution by using the Laplace transform method is rather straightforward and seems to be more easy than the use of the method of modal analysis based on eigenfunction expansions. Moreover, the Laplace transform method is nowadays well-described in elementary textbooks on partial differential equations (see for instance Ref. [8]). In forthcoming papers it will be shown that the Laplace transform method can efficiently and easily be applied to weakly perturbed or weakly nonlinear (axially moving) string equations.

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